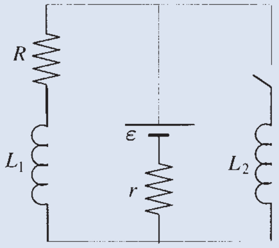
Physics Challenges for Teachers and Students

Solutions to May 2004 *Challenges*

Double Closure

Challenge: An electric circuit contains a battery with emf ε and internal resistance r, two coils with inductances L_1 and L_2, and a resistor R, connected as shown. On the diagram, all shown parameters are given. Initially, both switches are open. Switch S_1 is then closed. After a while, switch S_2 is closed. What is the total charge Q that passes through the resistor after S_2 is closed?



Solution: That charge is finite because the second inductor shorts out the battery so that the final voltage between the top and bottom of the diagram will be zero. It will take some time to reach that condition because the current in L_2 will approach its asymptotic value gradually. Let "I" be the current through L_1, and let "i" be the current through L_2. (Both currents are functions of time.) The voltages across the three vertical parts of the network must be equal:

$$V = RI + L_1\, dI/dt. \qquad (1)$$

$$V = \varepsilon - r(I + i). \qquad (2)$$

$$V = L_2\, di/dt. \qquad (3)$$

Combining Eqs. (1) and (3),

$$RI + L_1 dI/dt = L_2\, di/dt. \qquad (4)$$

It seems likely that V will decay exponentially. ($V = V_o e^{-kt}$) so the terms on the right sides of Eqs. (1) and (3) must decay in similar fashion:

$I = I_o e^{-kt}$, so $dI/dt = -kI_o e^{-kt}$,

and

$i = i_f(1 - e^{-kt})$ so $di/dt = k i_f e^{-kt}$.

By plugging those expressions into Eq. (4) and canceling the exponentials we find that $RI_o - kI_oL_1 = k i_f L_2$. Solving for the unknown constant, $k = RI_o/[I_oL_1 + i_fL_2]$ $= R/[L_1 + (i_f/I_o)L_2]$. But $I_o = \varepsilon/(R + r)$ and $i_f = \varepsilon/r$, so $i_f/I_o = (R + r)/r$.
Therefore $k = R/[L_1 + L_2(R + r)/r]$.

To check my solution I will see if it is consistent with Eq. (2):

$$V_o e^{-kt} = \varepsilon - r(I + i) = \varepsilon - r\{I_o e^{-kt} + i_f(1 - e^{-kt})\}$$
$$= (\varepsilon - r\, i_f) + r(i_f - I_o)e^{-kt}\ .$$

We already know that $(\varepsilon - r\, i_f) = 0$, so we can factor out the exponential:

$$V_o = r(i_f - I_o) = r[\varepsilon/r - \varepsilon/(R + r)]$$
$$= \varepsilon[1 - r/(R + r)] = \varepsilon[R + r - r]/(R + r)$$
$$= \varepsilon R/(R + r).$$

Since we know this is correct, the solution seems good. To find the total charge that passes through resistor R we must integrate I with respect to time, from $t = 0$ to ∞:

$$Q = \int I dt = I_o \int e^{-kt} dt = I_o/k$$
$$= [\varepsilon/(R + r)][L_1 + L_2(R + r)/r]/R$$
$$= (\varepsilon/R)[L_1/(R + r) + L_2/r].$$

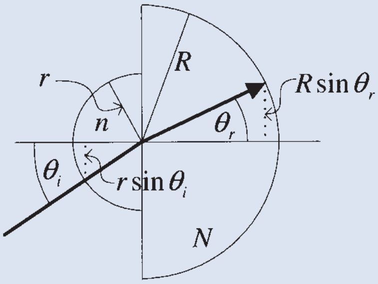
(Contributed by Art Hovey, Milford, CT)

Column Editor's Note: *We thank Leo H. van den Raadt (Heemstede, The Netherlands) for pointing out a typo in the originally posted solution.*

This revised pdf (posted July 12, 2004) replaces the May Solutions originally posted online.

Refraction in Action

Challenge: Two concentric hemispheres are made of different transparent materials. It is known that the ratio of the radii of the spheres equals the ratio of the corresponding indexes of refraction. A laser beam strikes one of the hemispheres as shown. Construct the exact path of the beam inside the hemispheres.



Solution: The small hemisphere at the left has index of refraction n and radius R. The large hemisphere at the right has index of refraction N and radius R. We are given that $r/R = n/N$. The ray strikes the plane separating the two hemispheres at incident angle θ_i and is refracted into the larger hemisphere at refracted angle θ_r. The incident ray strikes the small hemisphere's curved surface at a distance $r \sin \theta_i$ from the horizontal line in the figure. The refracted ray strikes the large hemisphere's curved surface at a distance $R \sin \theta_r$ from the horizontal line.

By Snell's law of refraction, $n \sin \theta_i = N \sin \theta_r$. Because $r/R = n/N$, it is easy to see that $r \sin \theta_i = R \sin \theta_r$. This means that the incident ray strikes the small hemisphere's surface at the same distance from the horizontal line that the refracted ray strikes the large hemisphere's surface.

Since the rays are coincident with radii of the hemispheres, reflections at the outer surfaces also are coincident with radii of the hemispheres. Not shown is the reflection of the incoming ray from the plane separating the two hemispheres. It would strike the curved surface of the small hemisphere at the same distance above the horizontal line that the incoming ray struck it below the horizontal line, that is, at distance $r \sin \theta_i$. Upon returning to the vertex at the center of both hemispheres, part of it would be reflected along the path of the original incoming ray and part would be refracted into the lower half of the right-hand hemisphere, where it would strike the curved surface at a distance $R \sin \theta_r$ below the horizontal line.

(Contributed by Charles B. Cameron,
U.S. Naval Academy, Annapolis, MD)

The Bulletproof Sandwich

Challenge: A thin plate of transparent plastic is embedded in a thick slab of glass. The index of refraction of the glass is $n = 1.50$; the index of refraction of the plate changes as shown in the diagram. A beam of light passes through glass and strikes the surface of the plastic plate. What maximum angle of incidence enables the beam to pass through the plate?

Solution: The problem in essence is one of total internal refraction. One must be sure that in traveling from the higher index of refraction to the lower index of refraction that total internal refraction does not occur. Let us consider the plastic to be "layered," i.e., many very thin layers of different index. This allows us a model by which we can construct Snell's law. At the top surface, Snell's law gives that $n_G \sin \theta_{\text{incident}} = n_2 \sin \theta_2$. The transition to the next layer would give $n_2 \sin \theta_2 = n_3 \sin \theta_3 \ldots$, so one notes that the original angle of incidence from the glass can be related to the second layer of the plastic. This procedure can be continued throughout the entire plastic. Hence, when the smallest index of refraction layer is reached, one can write $n_G \sin \theta_{\text{incident}} = n_p \sin \theta_p$. So, the maximal angle of incidence can be found by setting $\theta_p = 90°$ (condition for internal reflection to begin)…. Thus,

$$1.50 \sin \theta_{\text{incident}} = 1.20 \sin 90° \Rightarrow \sin \theta_{\text{incident}} = \frac{4}{5} \Rightarrow \theta_{\text{incident}} \approx 53.1°.$$

(Contributed by Michael C. Faleski,
Delta College, Midland, MI)

Several other readers also sent us correct solutions to the May *Challenges*. We would like to recognize the following contributors:

Phil Cahill (Lockheed Martin Corporation, Rosemont, PA)

Eugene P. Mosca (U.S. Naval Academy, Annapolis, MD)

Carl E. Mungan (U.S. Naval Academy, Annapolis, MD)

John Spokas (Benedictine University, Lisle, IL)

Leo H. van den Raadt (Heemstede, The Netherlands)

Silvano Sgrignoli (Istituto Tecnico e Liceo Scientifico-tecnologico "Giulio Natta," Bergamo, Italy)

We appreciate your submissions and hope to receive more solutions in the future.
Many thanks!

Note to Contributors:

As the number of submissions grows, we request that certain guidelines be observed in order to facilitate the process more efficiently:
– please email the solutions as Word files;
– please name the file "September04LSimpson" if— for instance—your name is Lisa Simpson, and you are sending the solutions to September 2004 Challenges;
– please state your name, hometown and professional affiliation in the file, not only in the email message.

Many thanks!

Please send correspondence to:

Boris Korsunsky
444 Wellesley St.
Weston, MA 02493-2631
korsunbo@post.harvard.edu