Physics Challenges for Teachers and Students

Solutions to March 2004 *Challenges*

A Disco Dance

Challenge: A small charged ball is hovering in the state of equilibrium at a height *h* over a large horizontal uniformly charged dielectric plate. What would be the acceleration of the ball if a disc of radius $r = 0.001h$ is removed from the plate directly underneath the ball?

Solution: Denoting the mass and charge of the ball by m and q, respectively, the initial force balance on the ball is $mg = qE$, where E is the magnitude of the electric field at the ball's location. If the plate has uniform surface charge density σ, then we can cancel the charge on a disc of radius r by superimposing uniform charge $-Q = -\sigma\pi r^2$ onto it. Since this disc is small, the electric field it creates at the ball's location is essentially that of a point charge, and hence the ball accelerates downward, under the action of the unbalanced electric force due to this field, with a magnitude a given by

$$ma = \frac{qQ}{4\pi\varepsilon_0 h^2} = \frac{(mg/E)(\sigma\pi r^2)}{4\pi\varepsilon_0 h^2}, \qquad (1)$$

where the two charges are substituted from the preceding two equations. Since the plate is large and uniformly charged, we can assume the electric field that the intact plate produces is given by the expression for an infinite plate, $E = \sigma/2\varepsilon_0$. Substituting this into Eq. (1) gives

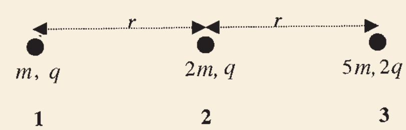
$$a = \frac{g}{2}\left(\frac{r}{h}\right)^2 = 4.9\ \mu\text{m/s}^2\ . \qquad (2)$$

Note that this result assumes that h is small compared to the horizontal distance from the ball to the closest edge of the plate, and this implies that the ball is initially merely in *neutral* rather than stable equilibrium. In fact, the small vertical gradient of the electric field due to the finite size of the plate (not to mention air currents and other small disturbances) could rather easily produce accelerations exceeding the tiny value given by Eq. (2).

(Contributed by Carl E. Mungan, U.S. Naval Academy, Annapolis, MD)

Three Is a Crowd

Challenge: Three small positively charged particles are held in place as shown. The masses and the charges of the particles are given, as well as the initial distance *r* between the "neighboring" particles. All three particles are released simultaneously. Find the kinetic energies of the particles when they are far from each other. Assume that the particles move along the same straight line. The particles are labeled 1, 2, and 3 as shown.



Solution: Let us denote the final velocities of the three particles by υ_1, υ_2, and υ_3. The initial energy of the system is all potential (relative to a reference at infinity) while the final energy is all kinetic, so that conservation of energy requires

$$\sum_{j>i}\frac{kq_iq_j}{r_{ij}} = \sum_i \tfrac{1}{2}m\upsilon_i^2 \;\Rightarrow\; \frac{8kq^2}{mr} = \upsilon_1^2 + 2\upsilon_2^2 + 5\upsilon_3^2\ , \qquad (1)$$

where $k = 1/4\pi\varepsilon_0$ is the Coulomb constant. Next let us explicitly consider the dynamics and kinematics during the separation of the charges. In their initial configuration, with rightward chosen to be positive, the net electric forces on the three charges are each given by the sum of the electric forces due to the other two charges,

$$ma_1 = \frac{kq^2}{r^2}\left[-\frac{1}{1}-\frac{2}{4}\right] \Rightarrow a_1 = -3\left(\frac{kq^2}{2mr^2}\right)$$

$$2ma_2 = \frac{kq^2}{r^2}\left[+\frac{1}{1}-\frac{2}{1}\right] \Rightarrow a_2 = -1\left(\frac{kq^2}{2mr^2}\right) \qquad (2)$$

$$5ma_3 = \frac{kq^2}{r^2}\left[+\frac{2}{4}+\frac{2}{1}\right] \Rightarrow a_3 = +1\left(\frac{kq^2}{2mr^2}\right)$$

where a denotes acceleration. Since the particles start out from rest, their displacements and velocities after the first small increment of time are in the ratio –3:–1:+1. In particular, if the (small) displacement of particle 3 is, say, $+x$ and we choose the origin at, say, the initial position of particle 1, then the positions of the three particles after this time increment are $-3x$, $r - x$, and $2r + x$, respectively. The key point to notice is that the distance between particles 1 and 2 (namely $r + 2x$) is *equal* to the distance between particles 2 and 3, just as it was initially. This means that the new accelerations, although slightly weaker, will *again* be in the ratio –3:–1:+1. Consequently the displacements and velocities will *always* be in the ratio –3:–1:+1! (Note that this result is consistent with conservation of linear momentum, which requires that the center of mass remain fixed, or equivalently that the sum of the momenta remains zero.) This implies that

$$\upsilon_1 = -3\upsilon_3 \text{ and } \upsilon_2 = -\upsilon_3. \qquad (3)$$

Substituting these two results into Eq. (1) implies that

$$\upsilon_3 = \sqrt{\frac{kq^2}{2mr}} \Rightarrow K_3 \equiv \frac{1}{2}5m\upsilon_3^2 = \frac{5kq^2}{4r}, \qquad (4)$$

and Eq. (3) then gives

$$K_1 \equiv \frac{1}{2}m\upsilon_1^2 = \frac{9kq^2}{4r} \text{ and } K_2 \equiv \frac{1}{2}2m\upsilon_2^2 = \frac{kq^2}{2r}. \qquad (5)$$

(Contributed by Carl E. Mungan, U. S. Naval Academy, Annapolis, MD)

Some Like It Hot

Challenge: Three identical containers, 1, 2, and 3, are filled with equal quantities of ice at 0°C and placed outside. Identical heating elements are placed inside each container. The elements are powered by different voltages: V_1 = 380 V, V_2 = 220 V, and V_3 = 110 V. In container 1, all ice melted in t_1 = 4 min. In container 2, all ice melted in t_2 = 20 min. How long would it take to melt all the ice in container 3? Assume that the resistance of the heating elements remains constant and that the temperature of each container is constant throughout its volume at any given instant.

Solution: The rate of electrical heating of the ice in a container is equal to the electrical power dissipated, V^2/R, where R is the resistance of the heating element. For containers 1 and 2, the ratio of electrical heating rates is thus (380 V/220 V)^2. However, it takes five, not three, times longer to melt the ice in container 2 compared to container 1. Hence, there must be a heat leak to the surroundings. Assuming that the containers were placed outside on a cold winter day for which the surroundings have an ambient temperature $T_0 <$ 0°C, Newton's law of cooling says that the rate of heat loss from the ice to the surroundings is $P_{\text{loss}} = k\Delta t$, which is a constant until all of the ice has melted.

In order to melt the ice in containers 1 and 2, we must have transferred the same net amount of heat to each,

$$\left(\frac{V_1^2}{R} - P_{\text{loss}}\right)t_1 = \left(\frac{V_2^2}{R} - P_{\text{loss}}\right)t_2 . \qquad (1)$$

Solving this equation using the given data, we find $P_{\text{loss}}R$ = (156 V)^2. The significance of this result is that we will just balance the heat loss to the surroundings if we run the heater at 156 V. Thus, if we apply only 110 V across the heating element, the container will *lose* heat and the ice will cool down instead of melting!

In fact, we can estimate to what final temperature T_{f} container 3 will cool down to reestablish thermal equilibrium between the rates of heat input and output. Our solution above can be rewritten using Newton's law of cooling as

$$k(0°\text{C} - T_0)\, R = (156\ \text{V})^2. \quad (2)$$

On the other hand, thermal equilibrium is achieved when

$$k(T_f - T_0)R = (110\ \text{V})^2. \quad (3)$$

Dividing Eq. (3) by (2) and simplifying implies that $T_f = T_0/2$. For example, if the outside temperature is –10°C, then container 3 will cool down exponentially and eventually level off at –5°C.

(Contributed by Carl E. Mungan, U. S. Naval Academy, Annapolis, MD)

Several other readers also sent us correct solutions to the March Challenges. We would like to recognize the following contributors:

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We appreciate your submissions and hope to receive more solutions in the future.

Note to contributors: As the number of submissions grows, we request that certain guidelines be observed, in order to facilitate the process more efficiently:

– please email the solutions as Word files;
– please name the file "May04HPotter" if—for instance—your name is Harry Potter, and you are sending the solutions to May 2004 Challenges;
– please state your name, hometown and professional affiliation in the file, not only in the email message.

Many thanks!

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